According to (3.3) loc. cit., if we transform from $(x^1, x^2, \ldots, x^n)$ to $(z^1, z^2, \ldots, z^n)$ the functions $\Gamma^i_{i\alpha}$ are transformed as follows:

$$\bar{\Gamma}^i_{i\alpha}(z) = \Gamma^i_{ij}(x)\frac{\partial x^j}{\partial z^\alpha} + \frac{\partial}{\partial z^\alpha}\left(\frac{\partial x^j}{\partial z^i}\right)\cdot\frac{\partial z^i}{\partial x^j}$$

$$= \Gamma^i_{ij}(x)\frac{\partial x^j}{\partial z^\alpha} + \frac{1}{\Delta}\frac{\partial \Delta}{\partial z^\alpha}$$

where Δ is the Jacobian, $|\partial x^j/\partial z^i|$ of the transformation. The last equations are the same as

$$\bar{\Gamma}^i_{i\alpha}(z) = \frac{\partial \log \gamma}{\partial x^j}\frac{\partial x^j}{\partial z^\alpha} + \frac{\partial \log \Delta}{\partial z^\alpha} = \frac{\partial}{\partial z^\alpha}\log \bar{\gamma}$$

where we are denoting by $\bar{\gamma}$ the function obtained by substituting the x's as functions of the z's in γ and multiplying by Δ. Hence the function γ is a scalar density and

$$\int \gamma\, dx^1\, dx^2 \ldots dx^n$$

may be taken as a definition of volume.

AFFINE GEOMETRIES OF PATHS POSSESSING AN INVARIANT INTEGRAL

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Read before the Academy, November 16, 1922

1. In the Riemann geometry volume is defined by the invariant integral $\int \sqrt{g}\, dx^1 \ldots dx^n$ where g is the determinant $|g_{ij}|$. If g' denotes the corresponding function when the coördinates are $x'^2, \ldots x'^n$, then

$$\sqrt{g'} = \sqrt{g}\,\Delta, \qquad (1.1)$$

where Δ is the Jacobian $\left|\frac{\partial x}{\partial x'}\right|$. When there exists for a geometry of paths a function g satisfying (1.1), we say that the geometry possesses an invariant integral, and g is called a *scalar density*. In a recent note (these PROCEEDINGS, **9**, p. 3) Professor Veblen showed that in an affine space for which $S_{ij} = 0$ a scalar density is defined by $\Gamma^\alpha_{\alpha i} = \partial \log \sqrt{g}/\partial x^i$; he calls such a space *equiaffine.* It is the purpose of this note to show that *a necessary and sufficient condition that a goemetry of paths possess an invariant integral is that S_{ij} be the curl of a covariant vector,* and to derive some consequences of this theorem.

2. Let Γ^i_{jk} be the functions appearing in the equations of the paths (these PROCEEDINGS, Feb., 1922), then the functions Γ'^i_{jk} for a set of coördinates x' are given by

$$\frac{\partial^2 x^p}{\partial x'^i \partial x'^j} + \Gamma^p_{qr}\frac{\partial x^q}{\partial x'^i}\frac{\partial x^r}{\partial x'^j} = \Gamma'^t_{ij}\frac{\partial x^p}{\partial x'^t}, \tag{2.1}$$

and the curvature tensor is defined by

$$B^p_{qrs} = \frac{\partial \Gamma^p_{qs}}{\partial x^r} - \frac{\partial \Gamma^p_{qr}}{\partial x^s} + \Gamma^p_{\alpha r}\Gamma^\alpha_{qs} - \Gamma^p_{\alpha s}\Gamma^\alpha_{qr}. \tag{2.2}$$

By definition we have

$$S_{ij} = B^\alpha_{\alpha ij} = \frac{\partial \Gamma^\alpha_{\alpha j}}{\partial x^i} - \frac{\partial \Gamma^\alpha_{\alpha i}}{\partial x^j}. \tag{2.3}$$

If equation (1.1) be differentiated, we have, in consequence of (2.1),

$$\begin{aligned}\frac{\partial\sqrt{g'}}{\partial x'^i} &= \frac{\partial\sqrt{g}}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial x'^i}\Delta + \sqrt{g}\,\Delta\frac{\partial^2 x}{\partial x'^i\partial x'^j}\frac{\partial x'^j}{\partial x^\alpha} \\ &= \Delta\sqrt{g}\left(\frac{\partial \log\sqrt{g}}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial x'^i} + \Gamma'^\alpha_{\alpha i} - \Gamma^\alpha_{\alpha q}\frac{\partial x^q}{\partial x'^i}\right),\end{aligned}$$

or, by means of (1.1),

$$\frac{\partial \log\sqrt{g'}}{\partial x'^i} - \Gamma'^\alpha_{\alpha i} = \left(\frac{\partial \log\sqrt{g}}{\partial x^j} - \Gamma^\alpha_{\alpha j}\right)\frac{\partial x^j}{\partial x'^i}. \tag{2.4}$$

From this equation it follows that

$$\partial \log\sqrt{g}/\partial x^j = \Gamma^\alpha_{\alpha j} - \varphi_j. \tag{2.5}$$

where φ_j is a covariant vector. The conditions of integrability of equations (2.5) are

$$\frac{\partial \Gamma^\alpha_{\alpha j}}{\partial x^i} - \frac{\partial \Gamma^\alpha_{\alpha i}}{\partial x^j} = \frac{\partial \varphi_j}{\partial x^i} - \frac{\partial \varphi_i}{\partial \varphi^j}, \tag{2.6}$$

that is, S_{ij}, as defined by (2.3), is the curl of a vector, φ.

Conversely, if S_{ij} is the curl of a vector, φ_i, we have equations of the form (2.6) and (2.5) in each coördinate system, x and x', and consequently equation (2.4) holds. If equations (2.1) be multiplied by $\frac{\partial x'^j}{\partial x^p}$ and summed for p and j, we obtain

$$\Gamma'^{\alpha}_{\alpha i} = \Gamma^{\alpha}_{\alpha q} \frac{\partial x^q}{\partial x'^i} + \frac{\partial \log \Delta}{\partial x'^i}.$$

By means of this relation, we obtain (1.1) from (2.4), and the theorem of § 1 is proved.

3. In a former paper (these PROCEEDINGS, Aug., 1922) the author considered spaces with corresponding paths and made the restriction in § 1 of that paper that s is the same for all paths. If this restriction be removed, the formulas (3.4) and (3.6) written in the form

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} + \delta^i_j \varphi_k + \delta^i_k \varphi_j \quad \left(\delta^i_j = {1 \text{ for } i = j \atop 0 \text{ for } i \neq j}\right), \qquad (3.1)$$

where φ_i is a covariant vector, give the necessary and sufficient relations between the Γ 's of two geometries of paths so that the paths are in one-to-one correspondence, and (3.7) gives the relations between s and $\bar{s}$ along corresponding paths. Equations (3.1) have been found by Weyl (*Gött. Nach.*, 1921), and also independently by Veblen (these PROCEEDINGS, Dec., 1922); they have interpreted them as the relations between the Γ's which yield the same paths in a space. Also they have remarked that each choice of the vector φ_i yields an affine space, whereas the paths define a projective space.

In my former paper it was shown that

$$\bar{S}_{ij} = S_{ij} + (n + 1)(\varphi_{ji} - \varphi_{ij}). \qquad (3.2)$$

From this equation it follows that if S_{ij} is the curl of a vector, and the vector $\varphi_i/(n + 1)$ is used in (3.1), then $\bar{S}_{ij} = 0$, that is, the space is equiaffine.

By definition the contracted curvature tensor R_{ij} is given by

$$R_{ij} = B^{\alpha}_{ij\alpha} \qquad (3.3)$$

From this it follows that

$$R_{ij} - R_{ij} = S_{ji} \qquad (3.5)$$

Hence the above result may be stated as follows:

Among the affine spaces possessing an invariant integral and having corresponding paths, one is equiaffine; for this space the contracted tensor is symmetric.

This result takes the place of the theorem stated in § 5 of my former paper, where an error was made in concluding that S_{ij} is the curl of a vector for any geometry of paths (cf. my note on this point in the *Bull. Amer. Math. Soc.*, Dec., 1922).

4. The contracted tensor for a Riemann space is symmetric. Consequently if in (3.1) we replace Γ^i_{jk} by their expressions as Christoffel symbols of the second kind for a Riemann space, the functions $\overline{\Gamma}^i_{jk}$ define an affine space possessing an invariant integral. Hence:

The spaces with paths corresponding to the paths of a Riemann space possess an invariant integral.

CLOSED CONNECTED SETS WHICH ARE DISCONNECTED BY THE REMOVAL OF A FINITE NUMBER OF POINTS

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THEOREM A. *Suppose k is a positive integer and M is a closed connected point set in Euclidean space of two dimensions such that*

(1) *if $P_1, P_2, \ldots P_k$ are any k distinct points of M, then $M - (P_1 + P_2 + \ldots + P_k)$ is disconnected.*

(2) *if $Q_1, Q_2, \ldots Q_{k-1}$ are any $(k - 1)$ distinct points of M, then $M - (Q_1 + Q_2 \ldots Q_{k-1})$ is connected.*

Under these conditions, M is a continuous curve.[1]

Proof.—Let us suppose that M is not connected im kleinen. Then there exists a point P belonging to M and a circle K with centre at P, such that within every circle whose centre is P there exists a point which does not lie together with P in any connected subset of M that lies entirely within K. Let $K_1, K_2 \ldots$ denote an infinite sequence of circles with centre at P and radius $r/2n$, where r is the radius of K. Let X_n denote a point within K_n such that X_n and P do not lie together in a connected subset of M which lies entirely within K. Let K' denote a circle with centre at P and radius $3r/4$. It follows with the use of a theorem due to Zoretti[2] that there is a closed connected set g_n, containing X_n and at least one point of K' not containing P and lying entirely within or on K'. It may easily be proved that there exist point sets $t_{n_1}, t_{n_2}, \ldots$ such that (1) for every i, t_{n_i} is a closed connected subset of M having at least one point on K' and at least one point on K_1 but no point within K_1 or without K', (2) for no values of i and j $(i \neq J)$ does t_{n_i} have a point in common with t_{n_j}. It follows that there exists an infinite sequence of integers $q_1, q_2, \ldots$ such that for every i, $q_{i+1} > q_i$ and a closed connected set t and a sequence of closed connected sets $k_{n_{q_1}}, k_{n_{q_2}} \ldots$ such that (1) for every i, $k_{n_{q_i}}$ is a subset of $t_{n_{q_i}}$, (2) each of